Complex Hadamard Matrices and Weighing Matrices

M. K. Singh[1] Sudha Singh[2] and D. K. Singh[3]

Abstract

It is shown that a weighing matrix can be obtained from any quaternary complex Hadamard matrix as well as from two suitable disjoint weighing matrices. It is also shown that Hadamard and complex Hadamard matrices can be obtained from complex weighing matrices. Weighing matrices can be regarded as ternary orthogonal code. Here rows of the matrix are taken as code words.

Keywords : Dephased Hadamard Matrix, Complex Hadamard Matrix, Butson Hadamard Matrix, Conference Matrix, Weighing Matrix, Disjoint Weighing Matrices, Complex Weighing Matrix.

1. Introduction

We begin with following definitions and basic facts:

(i) A **Hadamard matrix (or an H-matrix)** is an nxn matrix H with entries +1, -1 such that $HH^T = nI_n$, where I_n is the nxn identity matrix. If H-matrix of order n exists, n=1, 2 or 4t, where t is a positive integer.

[1]PG Department of Mathematics , Ranchi University, Ranchi - 834008, Jharkhand, India. Email : mithileshkumarsingh@gmail.com

[2]Research Scholar , BIT Sindri , Dhanbad - 828123, Jharkhand, India. Email : sudha_2k6@yahoo.com

[3]Department of Electronics and Communications BIT Sindri , Dhanbad - 828123, Jharkhand, India. Email : dksingh_bit@yahoo.com

(ii) Some generalizations of H-matrix:

(a) A complex H-matrix is an nxn matrix $H =[H_{ij}]$, where H_{ij} are complex numbers with $|H_{ij}| = 1$ for i, j = 1, 2, . . . ,n, satisfying $HH^* = nI_n$ where I_n is the identity matrix and H^* denotes the Hermitian transpose of H. A complex H-matrix is called dephased or normal if elements of its first row and column are 1.

(b) A Butson H-matrix (videButson[5]) is an nxn complex H- matrix with elements belonging to the set of m th roots of 1 and is denoted as BH(m,n). BH(4,n) containing all of ±1,±i is named as a quarternary complex Hadamard matrix by Horadam[11].

(c) Weighing matrix : A weighing matrix $W = W(n,w)$ of order n and weight w is an $n \times n$ (0,1, -1)-matrix such that $WW^t = wI$, where W^t stands for transpose of W.

A conference matrix of order n is a weighing matrix W(n,n-1)with zeroes on the diagonal.

Example : Consider the following matrix

$$\begin{pmatrix} 0 & + & + & + & + & + & + & + \\ + & 0 & + & + & - & + & - & - \\ + & + & + & - & + & - & - & 0 \\ + & + & - & + & - & - & 0 & + \\ + & - & + & - & - & 0 & + & + \\ + & + & - & - & 0 & + & + & - \\ + & - & - & 0 & + & + & - & + \\ + & - & 0 & + & + & - & + & - \end{pmatrix}$$

This is a symmetric weighing matrix W(8,7) but not a conference matrix.

(d) Complex weighing matrix

A matrix W of order n with entries $0,\pm1,\pm i$ is called a complex weighing matrix if $WW^{\dagger}=kI_n$, where $W^{\dagger}$ stands for Hermitian conjugate of W and k is a positive integer.

(iii) disjoint weighing matrices

Two real or complex weighing matrices $W_1=[\ m_{ij}\ ]$ and $W_2=[\ n_{ij}\ ]$, will be called disjoint if

$$\begin{cases} m_{ij} \neq 0 \Rightarrow n_{ij} = 0 \\ n_{ij} \neq 0 \Rightarrow m_{ij} = 0 \end{cases}$$

For the introduction to Hadamard matrices we refer to Hall[9], Hedayat, et al[10] Agaian[1] and to Horadam[11] for their generalizations. For tables of H-matrices vide Hall[9] and Seberry's website[16].

Turyn[17] and. Wallis[18] constructed infinite families of complex Hadamard matrices . Circulant Complex Hadamard matrices were studied by, Arasu, et al [3]. Gysin and Seberry [12] forwarded methods of constructing W(4n, 4n-2) and W(4n, 2n-1) using conference matrices, Golay sequences and cyclotomy. Gysin and Seberry [13] obtained weighing matrices through linear combinations of generalized cosets and Craigen[6] studied weighing matrices of large weights.. Arasu, et al [2] constructed Circulant weighing matrices of weight 2^{2t}. The present paper was motivated by the half-full conjecture quoted by Gysin and Seberry[12] as well as Craigen[7] which states that a weighing matrix W(4n, 2n) exists for every n.. This conjecture have been verified for 2n≤ 212(vide Craigen[7]) The theorem 2 of this paper shows that this conjecture is implied by another conjecture quoted by Wallis in [18] which states that a Butson Hadamard matrix BH(4, 2n) exists for every n.

Hadamard matrices from weighing matrices were constructed by Craigen and Kharaghani [8]. For Complex weighing matrices we refer to Berman [4] and Seberry[14,15].

Complex Hadamard matrices have applications in quantum information theory and quantum tomography.

Recently weighing matrices have been found much beneficial to engineers working with satellite and digital communications. They have been found to have many similarities with perfect ternary arrays, and these arrays have been implemented in our digital communications.

The purpose of the paper is to forward simple constructions for some of these matrices so that they can be used by engineers.

2 (i) Construction of a new weighing matrix from two disjoint weighing matrices:

Theorem 1: If W_1 and W_2 are disjoint nxn weighing matrices of weights k_1 and k_2 respectively, then

$$W=\begin{pmatrix} W_1+W_2 & W_1-W_2 \\ W_1^t-W_2^t & -W_1^t-W_2^t \end{pmatrix}$$

is a weighing matrix $W(2n,\ 2(k_1+k_2))$.

Proof : We have

$$WW^{\dagger}=\begin{pmatrix} W_1+W_2 & W_1-W_2 \\ W_1^t-W_2^t & -W_1^t-W_2^t \end{pmatrix}\begin{pmatrix} W_1^t+W_2^t & W_1-W_2 \\ W_1^t-W_2^t & -W_1-W_2 \end{pmatrix}$$

$$=\begin{pmatrix} 2(W_1W_1^t+W_2W_2^t) & (W_1^2-W_2^2)-(W_1^2-W_2^2) \\ 0 & 2(W_1W_1^t+W_2W_2^t) \end{pmatrix}$$

$$=\begin{pmatrix} 2(k_1I_n+k_2I_n) & 0 \\ 0 & 2(k_1I_n+k_2I_n) \end{pmatrix}$$

$$=2(k_1+k_2)I_{2n}$$

Therefore, $W = W(2n, 2(k_1+k_2))$.

Corollary: If $k_1+k_2=n$, then W(2n, 2n) is a Hadamard matrix of order 2n.

(ii) Construction of weighing matrix from the Butson Hadamard matrix B(4, 2n)

Theorem 2 For a Butson Hadamard matrix B(4,2n), there exists a weighing matrix W(4n, 2n) of order 4n and weight 2n..

Proof: Let S be the algebra of 2x2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ over the real field R. Let C be algebra of complex numbers over R. Then the mapping $\phi : C \to S$ which carries a+ib into the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is an algebra isomorphism which gives faithful representation of complex number a+ib by the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

In this representation 1 can be represented by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and i can be represented by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

This isomorphism can be extended to the algebra of complex matrices to the algebra of real matrices by the mapping f which carries a complex matrix M= $[a_{jk}+ib_{jk}]$ of order 2n to a real matrix f(M)= $[\phi(a_{jk}+ib_{jk})]$ of order 4n.

i.e. f(M) can be obtained from M by replacing the entry a+ib in M by the block $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

The isomorphism takes into account not only addition, usual product and multiplication by scalars but also unary operation of conjugation. Conjugate of (a+ib) corresponds to transpose of the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ under the isomorphism.

In particular, if we replace 1 by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, -1 by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, i by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and –i by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in B=BH(4,2n). we get a block matrix W of order 4n where blocks are the above 2x2 matrices.. Hence WW^t is the matrix whose diagonal blocks are 2n $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and non-diagonal blocks are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

i.e. $WW^t = 2nI_{4n}$.

Thus W is the required weighing matrix.

Illustration : Consider the matrix BH (4, 4) =

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$ which produces

The weighing matrix W(4, 8) =

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 \end{bmatrix}$$

Gysin and Seberry [12] made nearly half-full weighing matrix conjecture which states that a weighing matrix W (4n, 2n-1) exists for every positive integer n. They constructed a finite number of such weighing matrices. Our technique is capable of constructing some infinite classes of weighing matrices by simple replacements, supporting half-full conjecture.

3. Complex H-Matrix From Complex Weighing Matrices

Theorem 3: If there are two disjoint weighing matrices W_1, W_2 with entries $0, \pm 1, \pm i$ of order n and weights k_1, k_2 such that $k_1 + k_2 = n$, then

$$\begin{pmatrix} W_1 + W_2 & W_1 - W_2 \\ W_1^t - W_2^t & -W_1^t - W_2^t \end{pmatrix}$$ is a complex Hadamard matrix of order 2n if W^t stands for Hermitian conjugate of W.

Proof is analogous to that of theorem 1.

4. Conclusion

It is shown that a weighing matrix can be obtained from any quaternary complex Hadamard matrix as well as from two suitable disjoint weighing matrices. Since infinitely many BH(4,2n) are known (videTuryn[17]and Wallis[18]), it follows that infinitely many weighing matrices W (4n, 2n) can be constructed by theorem 2. It is also shown that Hadamard and complex Hadamard matrices can be obtained from certain disjoint weighing and complex weighing matrices respectively.

References

[1] Agaian S.S, *"In Hadamard Matrices and their Applications"*, Vol. 1168 of Lecture Notes in Mathematics. Springer-Verlag, Berlin-Heidelberg-New York, 1985.

[2] Arasu. K.T, Leung.K.H, Ma. S. L, Nabavi. A and Choudhary, D.K.R, *"Circulant Weighing matrices of weight 2^{2t}.Journal of Designs, Codes and Cryptography"*, Springer Neitherlands, Vol.41, No.1, PP. 111-123, Oct 2006.

[3] Arasu. K. T, W. de Launey and S. L. Ma, *"On circulant complex Hadamard matrices"*, Des. Codes Cryptogr. 25, 123-142, 2002.

[4] Berman .G, *"Families of generalized weighing matrices"*, Canadian J. Math. 30, PP.1016–1028,1978.

[5] Butson .A. T, *"Generalised Hadamard matrices"*, Proc. Amer. Math. Soc., 13 , PP. 894-898, 1962.

[6] Craigen .R, *"The structure of weighing matrices having large weights"*, Springer, Neitherlands, Vol.5, No.3, May.1995.

[7] Craigen. R, *"Weighing matrices and weighing designs in CRC Handbook of Combinatorial Designs (Eds: J. Dinitz and C Colburn"*, CRC Press, 1996).

[8] Craigen. R and H. Kharaghani, *"Hadamard matrices from weighing matrices vis signed groups"*, Des. Codes Cryptogr., 12, 49-58, 1997.

[9] Hall. Jr. M , *"Combinatorial Theory"*, Blaidell, Waltham, Mass, 1986.

[10] Hedayat A.S, N.J.A. Sloane and J. Stufken, *"Orthogonal Arrays, Theory and Applications"*, Springer, New York, 1999.

[11] Horadam. K.J, *"Hadamard matrices and their applications"*, Princeton University press, New Jersey, 2007.

[12] Marc Gysin and Jennifer Seberry, *"Construction methods for weighing matrices of order 4n and weight 4n-2, 2n-1 and n using elementary properties of cyclotomy"*, In Conference on Combinatorial Mathematics and Combinatorial Computing, Auckland, 1994.

[13] Marc Gysin and Jennifer Seberry, *"New weighing matrices through linear combinations of generalized cosets"*, In Conference on Combinatorial Mathematics and Combinatorial Computing, University of Technology, Sydney, July, 1996.

[14] Seberry Jennifer, *"Some remarks on generalized Hadamard matrices and theorems of Rajkundlia on SBIBDs, Combinatorial Mathematics VI"*, Lecture Notes in Mathematics, Vol. 748, Springer-Verlag, Berlin- Heidelberg-New York, 154-164, 1979.

[15] Seberry. J, *"complex weighing matrices and orthogonal designs"*, Ars Combinatoria ,V9, 149-162, 1980.

[16] Seberry .J, Website http://www.uow.edu.au/ jennie

[17] Turyn.R.J, *"Complex Hadamard matrices, Structures and their Applications"*, Gordon and Breach, New York, 1970.

[18] Wallis.J, *"Complex Hadamard matrices, Linear and multilinear algebra"*, Vol.I,257-272, 1993.

Author's Biography

Prof Mithilesh Kumar Singh is University Professor at Ranchi University Ranchi. He is actively arch in various disciplines which includes Coding theory and Network Security, Mathematical modelling, Discrete and Combinatorial Mathematics and Mathematical Biology. He has *Published* more than 30 papers in International and National Journals. He has also visited IISc, Bangalore and JNU, New Delhi as scientist during the period 1996-98. Presently he is Director MCA Program and Head of PG Department of Mathematics at Ranchi University, Ranchi.

Sudha Singh received her M.Tech. degree in computer science from Birla Institute of Technology,Mesra,Ranchi,India, in 2002and was placed First Class First. She obtained her Ph.D. degree in Computer Science and Engineering at the BIT Sindri,(VBU),India. She has more than 10 publications. Her research interest is in recent advances in Data communication, network security, coding theory, signal processing and combinatorics.

Dr. Dharmendra K Singh is presently working as Head, Department of Electronics and Communication & Information Technology, BIT Sindri, Dhanbad. He has more than 20 years of teaching experience. He is heading the department of Electronics and Communication & Information technology since 2002. He is instrumental in starting the curriculum on information technology. He has published more than 30 papers in journals and conferences. He has already supervised 01 thesis at doctoral level in computer Science & Engg and 05 research scholars are presently enrolled under him for their doctoral degree. The area of research he works are Coding theory, cryptography, optical Amplifiers , Photonic Crystal Fibers, e-Governance and technical education Planning.