A Simple Algebraic approach for Model Reduction of Multivariable Linear Time Invariant Systems

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Abstract

This paper presents a Simple Algebraic scheme for obtaining a second order approximant of a given absolutely stable higher order Linear Time invariant Multivariable system available either in state space or transfer function form. A guideline has been suggested for identifying an expansion point 'a' based on centroid concept for the purpose of order reduction. Two point expansion scheme in the s-domain between s = 0 and s= a is used to get a suitable second order system for the individual **SISO** systems that constitute the given **MIMO** system. A new algebraic scheme is proposed for obtaining the common denominator of the required second order **MIMO** model. Based on this common denominator and the Transient and Steady state gains of the original system, the individual second order **SISO** models are reconstructed and the final transfer function matrix of the second order **MIMO** model is declared. For discrete systems, a linear transformation can be used to analyze the problem in the s-domain. The proposed methodology is illustrated with a numerical example taken from the literature.

Keywords: Multivariable systems, State Space, Transfer function Matrix, Second order systems, Root Loci Centroid, Continued Fraction Technique, Linear transformation.

1. Introduction

Due to the increasing complexity of systems that must be controlled and in the interest to achieve optimum performance, the importance of control system engineering has grown in the past decade. As the systems become more complex, the interrelationship of many controlled variables must be considered in the control scheme. This leads to control systems that have more than one feedback loop. Such systems are nontrivial and are much more challenging than single-loop control systems. Multiple control loops are required whenever a plant has multiple sensors and actuators. While many single-loop concepts hold in principle in the multi-loop case, the technicalities are much

more involved. The performance benefits of multi-loop control are often far more than one would expect from a collection of single-loop controllers. Such multivariable control is essential in Multi-input Multi-output (MIMO) systems. They use measurements of several output variable and may involve manipulation of more than one input variable. The computer control systems used to control the fuel injectors and spark timing of automobiles are excellent examples of Multivariable control systems.

The exact analysis and synthesis of a high order multivariable system is often difficult and possibly not desirable on economic and Computational considerations.

Hence, it is necessary to obtain a lower order model so that it may be used for simulating the system. It is essential that the obtained lower order model preserve the important characteristics of the original system. This will minimize variations. During design and realization of suitable control system components to be attached to the original system.

Several model reduction methods **[1-5]** have been developed during the past three decades, which may be broadly placed into two categories. The first category includes Time domain methods **[6-9]** and the second category consists of Frequency domain methods **[10-14]**. Each method has its merits and applications. For time domain method, it is mandatory that the original higher order system is represented in state space form. In the frequency domain method, a Linear Time Invariant q input and p output system is to be given in $p \times q$ transfer function matrix.

In this paper a computationally simple algebraic scheme is proposed for obtaining a second order transfer function matrix for a given absolutely stable higher order multivariable system in the frequency domain. If the original system is represented in state space form, it is converted to transfer function matrix form using standard algorithms [15]. The rest of the paper is organized as follows. **Section 2** defines the problem and **Section 3** outlines the steps of an algorithm to implement the proposed methodology. Numerical illustrations are presented in **section 4**, followed by discussion and conclusion in **section 5.**

1. Problem Statement

Consider an n[th] order linear time invariant dynamic multivariable system with q inputs and p outputs described in time domain by state space equations given as:

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}\right\} \qquad \textbf{(2.1)}$$

where $\dot{x}$ is n dimensional state vector, u is q dimensional control vector and y is p dimensional output vector with $p \leq n$ and $q \leq n$. Also, A is $n \times n$ system matrix, B is $n \times q$ input matrix and C is $p \times n$ output matrix.

Alternatively, the above system may be described in frequency domain by the transfer matrix $G(s)$ of order $p \times q$ given as:

$$G(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{n-1} A_i s^i}{\sum_{i=0}^{n-1} a_i s^i + s^n} \qquad \textbf{(2.2)}$$

where $N(s)$ is the numerator matrix polynomial and $D(s)$ is the common denominator of the original system. Also, A_i are the constant matrices of order $p \times q$ and A_{n-1} may be null matrix and a_i are constants.

Irrespective of the form represented in equation **(2.1)** or **(2.2)** (time or frequency domain) of the original system $G(s)$, the problem is to find a k^{th} order reduced model $R^k(s)$, where $k < n$ in the following form represented by equation **(2.3)**, such that the reduced model retains the important characteristics of the original system and approximates its response as closely as possible for the same type of inputs.

$$R^k(s) = \frac{N^k(s)}{D^k(s)} = \frac{\sum_{i=0}^{k-1} B_i s^i}{\sum_{i=0}^{k-1} b_i s^i + s^k}$$

(2.3)

where $N^k(s)$ and $D^k(s)$ are respectively, the numerator matrix polynomial and the common denominator of the reduced model. Also, B_i are the constant matrices of the same order $p \times q$ and b_i are constants.

1. Algorithm for the Proposed Scheme

In this paper, assuming that the original system is described by equation **(2.2)**, our proposed scheme finds a suitable second order model of the form represented in equation **(2.3)** for k = 2.

The steps involved are as follows:

1. With the common denominator *D(s)* of the original system, the transfer function of the system can be represented in the matrix form:

$$G(s) = \frac{\begin{bmatrix} N_{11}(s) & N_{12}(s) & \dots & N_{1q}(s) \\ N_{21}(s) & N_{22}(s) & \dots & N_{2q}(s) \\ \dots & \dots & \dots & \dots \\ N_{p1}(s) & N_{p2}(s) & \dots & N_{pq}(s) \end{bmatrix}}{D(s)}$$

(3.1)

where $N_{ij}(s) = \sum_{k=0}^{n-1} A_k(i,j)s^k$

$i = 1,2,\dots,p$

$j = 1,2,\dots,q$

2. For each $G_{ij}(s)$ do the following:

2.1 From equation (2.2), we can write

$$G_{ij}(s) = N_{ij}(s) / D(s)$$

(3.2)

2.2 Compute

(i) $|sumofpoles| = a_{n-1}/1$

(ii) $|sumofzeroes| = A_{n-2}(i,j)/A_{n-1}(i,j)$

(iii) $Numberofpoles = n$

(iv) $Numberofzeroes = n-1$

2.3 Compute four expansion points using the following guideline:

$$a_i = \frac{|sumofpoles| \pm |sumofzeroes|}{Numberofpoles \pm Numberofzeroes}$$

(3.3)

2.4 Compute a single expansion point a as, 2.4

$$a = \frac{\sum_{i=1}^{4} a_i}{4}$$

(3.4)

2.5 Using continued fraction expansion technique about two points $s = 0$ and $s = a$ [16], obtain a second order model for the transfer function $N_{ij}(s)/D(s)$ as,

$$G_{ij}^2(s) = \frac{N_{ij}^2(s)}{D_{ij}^2(s)}$$

$$= \frac{c_{ij(0)} + c_{ij(1)}s}{d_{ij(0)} + d_{ij(1)}s + s^2}$$

(3.5)

where $c_{ij(0)}$, $c_{ij(1)}$, $d_{ij(0)}$ and $d_{ij(1)}$ and are constants. Without loss of generality, the coefficient of term in the denominator can be arrived at as 1.

2.6 Compute the Transient gain(Tg_{ij}) and Steady state gain (Sg_{ij}) of $G_{ij}(s)$ as:

$$Tg_{ij} = A_{n-1}(i,j)/1$$

(3.6)

$$Sg_{ij} = A_0(i,j)/a_0$$

(3.7)

3 Determine the common denominator $D^2(s)$ as the arithmetic mean of the corresponding coefficients of

each $D_{ij}^2(s)$ which is mathematically represented by

$$D^2(s) = d_0 + d_1 s + s^2 \qquad (3.8)$$

where

$$d_k = \frac{\sum_{i=1}^{p}\sum_{j=1}^{q} d_{ij(k)}}{p+q} \quad k = 0,1 \qquad (3.9)$$

4 Reconstruct the numerators of each $G_{ij}^2(s)$ as,

$$N_{ij}^2(s) = (d_0 \times Sg_{ij}) + Tg_{ij} \qquad (3.10)$$

so that the characteristics of the original higher order system are maintained in the proposed second order model.

5 The transfer function matrix of the second order system can now be represented as,

$$G_{ij}^2(s) = \frac{\begin{bmatrix} N_{11}^2(s) & N_{12}^2(s) & \dots & N_{1q}^2(s) \\ N_{21}^2(s) & N_{22}^2(s) & \dots & N_{2q}^2(s) \\ \dots & \dots & \dots & \dots \\ N_{p1}^2(s) & N_{p2}^2(s) & \dots & N_{pq}^2(s) \end{bmatrix}}{D^2(s)} \qquad (3.11)$$

Note:

i. Further, if required an expansion point, *a2* = *1/a* can also be considered for model reduction.

ii. For Linear time invariant multivariable discrete systems, a linear transformation of z = s + 1 is to be used in step 1 to obtain the equivalent transfer function matrix in the s-domain. In step 5, the reverse transformation of s = z − 1 is to be used for declaring the transfer function matrix of the second order model in the z-domain.

4. Numerical Illustration

Linear Time Invariant Continuous MIMO System

Consider the sixth order system[17,18] described by the transfer function matrix

$$[G(s)] = \begin{bmatrix} \frac{2(s+5)}{(s+1)(s+10)} & \frac{(s+4)}{(s+2)(s+5)} \\ \frac{(s+10)}{(s+1)(s+20)} & \frac{(s+6)}{(s+2)(s+3)} \end{bmatrix} \qquad (4.1)$$

1. The common denominator *D(s)* of sixth order system is,

$$D(s) = (s+1)(s+2)(s+3)(s+5)(s+10)(s+20) = s^6 + 41s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000 \qquad (4.2)$$

2. Now, *G(s)* can be represented as

$$[G(s)] = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \qquad (4.3)$$

where

$$G_{11}(s) = \frac{2s^5 + 70s^4 + 762s^3 + 3610s^2 + 7700s + 6000}{D(s)} \qquad (4.4)$$

$$G_{12}(s) = \frac{s^5 + 38s^4 + 459s^3 + 2182s^2 + 4160s + 2400}{D(s)} \qquad (4.5)$$

$$G_{21}(s) = \frac{s^5 + 30s^4 + 331s^3 + 1650s^2 + 3700s + 3000}{D(s)} \qquad (4.6)$$

$$G_{22}(s) = \frac{s^5 + 42s^4 + 601s^3 + 3660s^2 + 9100s + 6000}{D(s)} \qquad (4.7)$$

3. It is observed that for $G_{11}(s), G_{12}(s), G_{21}(s)$ and $G_{22}(s)$, the Number of Poles is 6 and Number of Zeroes is 5

4. The values of | Sum of Poles |, | Sum of Zeroes |, Expansion Points ('*a*', '*1/a*'), Transient gain (Tg) and Steady state gain (Sg) required for the algorithm are tabulated in **Table 4.1:**

Table 4.1 Parameters of original higher order MIMO system represented in equation (4.1)

Parameter	$G_{11}(s)$	$G_{12}(s)$	$G_{21}(s)$	$G_{22}(s)$
| Sum of Poles |	41	41	41	41
| Sum of Zeroes |	35	38	30	42
'a'	22.3636	22.3636	21.8181	22.909
'1/a'	0.0447	0.0447	0.0458	0.0436
(Tg)	2	1	1	1
(Sg)	1	0.4	0.5	1

5. The transfer functions of the second order models $R_{11}(s), R_{12}(s), R_{21}(s)$ and $R_{22}(s)$ corresponding to $G_{11}(s), G_{12}(s), G_{21}(s)$ and $G_{22}(s)$ obtained by Continued Fraction technique using Two point expansion scheme between s =0 and s = 1/a are:

$$R_{11}(s) = \frac{1.8892s + 8.7478}{s^2 + 9.7622s + 8.7478} \quad \textbf{(4.8)}$$

$$R_{12}(s) = \frac{0.9903s + 3.6465}{s^2 + 6.5780s + 9.1162} \quad \textbf{(4.9)}$$

$$R_{21}(s) = \frac{0.8778s + 7.4139}{s^2 + 15.8419s + 14.8278} \quad \textbf{(4.10)}$$

$$R_{22}(s) = \frac{1.0284s + 5.6552}{s^2 + 4.7986s + 5.6552} \quad \textbf{(4.11)}$$

6. Using the denominators of $R_{11}(s), R_{12}(s), R_{21}(s)$ and $R_{22}(s)$ represented by equations (4.8) through (4.11), the common denominator for the second order MIMO system is obtained by computing the Arithmetic Mean of the corresponding coefficients, which is represented as:

$$D^2(s) = s^2 + 9.2451s + 9.5866 \quad \textbf{(4.12)}$$

7. Using the Transient Gain and Steady State Gain of $G_{11}(s), G_{12}(s), G_{21}(s)$ and $G_{22}(s)$ from Table 4.1 and $D^2(s)$, the transfer functions represented in equations (4.8) through (4.11) can be reconstructed as:

$$R_{11}(s) = \frac{2s + 9.5866}{s^2 + 9.2451s + 9.5866} \quad \textbf{(4.13)}$$

$$R_{12}(s) = \frac{s + 3.8346}{s^2 + 9.2451s + 9.5866} \quad \textbf{(4.14)}$$

$$R_{21}(s) = \frac{s + 4.7933}{s^2 + 9.2451s + 9.5866} \quad \textbf{(4.15)}$$

$$R_{22}(s) = \frac{s + 9.5866}{s^2 + 9.2451s + 9.5866} \quad \textbf{(4.16)}$$

8. The second order MIMO model in transfer function matrix is:

$$G^2(s) = \begin{bmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{bmatrix}$$

$$= \frac{1}{D^2(s)} \begin{bmatrix} 2s + 9.5866 & s + 3.8346 \\ s + 4.7933 & s + 9.5866 \end{bmatrix} \quad \textbf{(4.17)}$$

9. The unit step time responses of the original higher order system represented in equation **(4.1)**, the proposed second order model given in equation **(4.17)** and the models obtained by the other methods from the literature **[17-18]** are shown as in **Fig 4.1(a) – 4.1(d).**

10. The Integral Square Error '*J*' for the proposed scheme and the other schemes are tabulated in **Table 4.2**

Table 4.2 Comparison of Integral Square Error for Illustration

Model Reduction Method	**Cumulative Error Index *J* for 10 Secs**			
	$G_{11}(s)$	$G_{12}(s)$	$G_{21}(s)$	$G_{22}(s)$
Proposed	**0.0576**	**0.0384**	**0.0281**	**0.1222**
R.Prasad [19]	0.1676	0.0955	0.0307	0.1970
J.Pal [20]	0.3068	3.8578	0.7160	0.2168
Routh	0.2301	0.0887	0.0468	0.2114

From the unit step time responses shown in Fig 4.1(a) – 4.1(d), it is observed that the characteristics of the proposed second order MIMO model is in close agreement with that of the original higher order MIMO system. Also, the cumulative error index '*J*' is minimum when compared with that of the other methods[14,19,20] taken from literature.

5. Discussion and Conclusion

In this paper a simple algebraic scheme for model reduction of Linear Time Invariant MIMO Systems has been presented. A guideline for identifying an expansion point has been suggested. The expansion point '*a*' is computed using the number of poles, zeroes and their absolute sums. This is related to the centroid of a system and hence provides a scientific choice for the expansion point. In case this expansion point is not suitable, an alternate expansion point of '*1/a*' has been suggested. Continued Fraction Expansion technique about two points $s = 0$ and $s = a$ is used to obtain second order models for the transfer functions of the original higher order MIMO system expressed in transfer function matrix form with common denominator. The denominators of the obtained second order models are combined to form the common

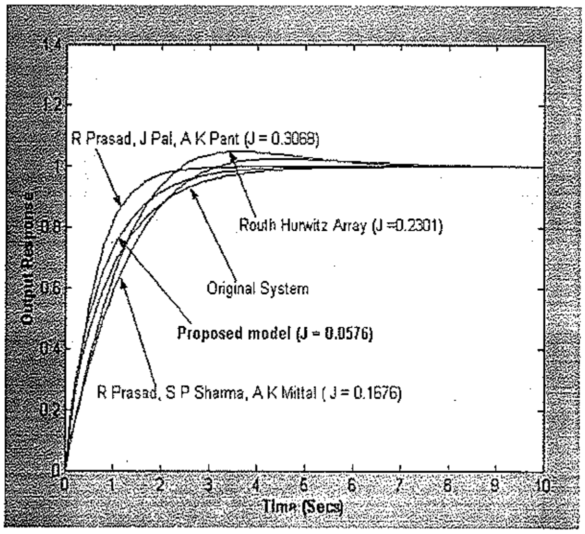

Fig 4.1(a) Unit step time response of $G_{11}(s)$

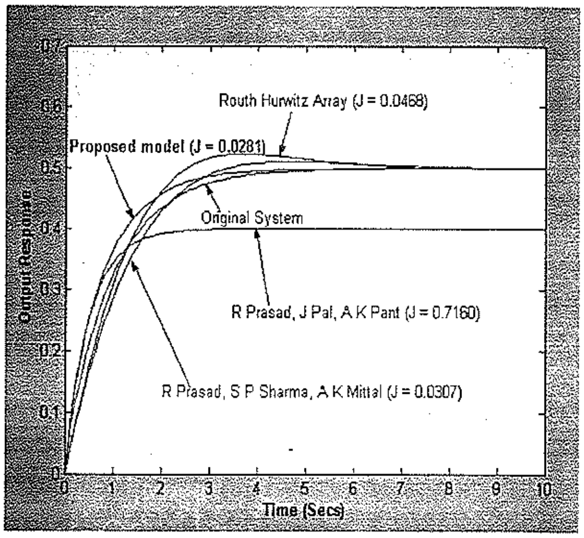

Fig 4.1(b) Unit step time response of $G_{12}(s)$

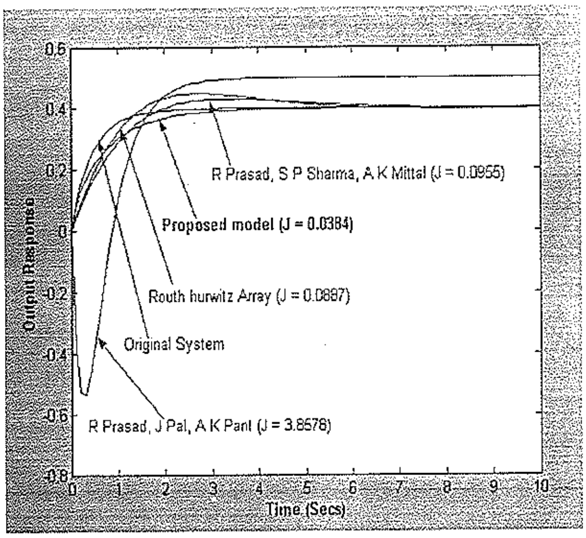

Fig 4.1(b) Unit step time response of $G_{12}(s)$

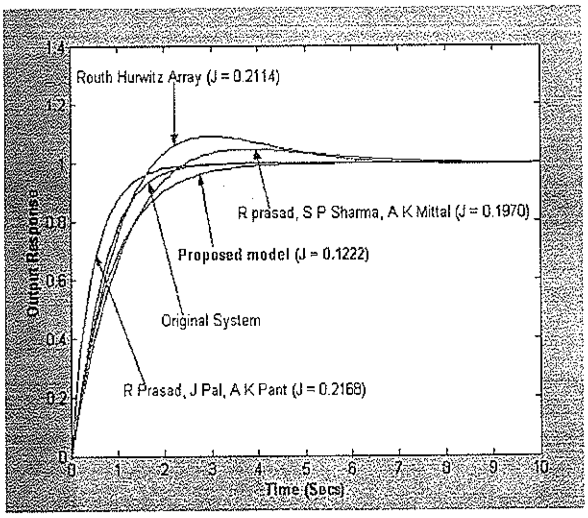

Fig 4.1(d) Unit step time response of $G_{22}(s)$

denominator of the proposed second order MIMO model. This is done by computing the Arithmetic Mean of the corresponding coefficients of the terms involved in the individual quadratic polynomials. This is a good approximation as we are computing the average of the sum of the roots and product of the roots of the characteristic equations of the individual second order models. Further, Transient and Steady State Gains of the original MIMO system are used to declare the transfer function matrix of the required second order MIMO model. The proposed scheme is applicable for both Linear Time Invariant Continuous and Discrete Systems. From the unit step time response of the numerical illustration, it is observed that the proposed second order MIMO model maintains the characteristics of the original higher order system with minimum value for the cumulative error index.. The obtained second order model can be further used for designing suitable state space observers and controllers for the given higher order MIMO system. The proposed methodology can be easily implemented on any digital computer.

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